ON POSSIBLE SIMPLIFICATIONS OF THE EQUATIONS OF A FULLY IONIZED TWO-TEMPERATURE PLASMA

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The system of equations, describing the behavior of a two-temperature (ion temperature unequal to electron temperature) fully-ionized plasma [1], appears very complicated for the solution of concrete problems.

In the equation of motion, in addition to ion viscosity, there also enters the electron viscosity, which is usually neglected for a one-temperature plasma (ion temperature equal to electron temperature). The problem is strongly complicated by the anisotropy of the transport coefficients, which we must consider when $\omega_i\tau_i \geqslant 1$, $\omega_e\tau_e \geqslant 1$; here ω_e and ω_i denote the cyclotron frequencies of the electrons and ions, and τ_e and τ_i denote the "mean collision times" of the electrons and ions. Thus, instead of two thermal conductivity coefficients for the ions and electrons and an electrical conductivity, we must now write three thermal conductivities for the electrons, three for the ions, and three electrical conductivities. Viscosity in this case is defined by five coefficients for the electrons, five for the ions, and ten second-rank viscosity tensors for the ions and electrons.

In the present paper, we shall estimate the different terms in the equations in order to simplify them. We write the values of the various critical parameters, for which various simplifications may be made (i.e. neglecting the anisotropy in the transport coefficients, the viscosity of the electrons in the equation of motion or in Ohm's law, etc.). It turns out that for certain values of these parameters, the viscous terms must be kept in the Ohm's law, so that Ohm's law becomes a differential equation and not just an algebraic relation. Also possible are cases when electron viscosity should be considered in the equation of motion while the ion viscosity may be neglected, etc. Most of these phenomena are connected with two-temperature plasmas and do not appear in one-temperature plasmas.

1. The system of equations for a fully ionized two-temperature plasma.

We shall consider a fully ionized plasma, consisting of two components, ions and electrons. For definiteness, we assume that the ions are singly ionized. In [1], the following system of equations was obtained describing the behavior of such plasmas:

$$\frac{\partial n_e}{\partial t} + \operatorname{div} n_e \mathbf{v}_e = 0, \quad \frac{\partial n_i}{\partial t} + \operatorname{div} n_i \mathbf{v}_i = 0 \quad \left(\frac{d_j}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}_j \nabla)\right) \quad (1.1)$$

$$m_e n_e \frac{d_e v_e^{\alpha}}{dt} = -\frac{\partial p_e}{\partial x_\alpha} - \frac{\partial \pi_e^{\alpha\beta}}{\partial x_\beta} - en\left(E_\alpha + \frac{1}{c}(\mathbf{v}_e \times \mathbf{H})_\alpha\right) + R_\alpha$$
$$m_i n_i \frac{d_i v_i^{\alpha}}{dt} = -\frac{\partial p_i}{\partial x_\alpha} - \frac{\partial \pi_i^{\alpha\beta}}{\partial x_\beta} + en\left(E_\alpha + \frac{1}{c}(\mathbf{v}_i \times \mathbf{H})_\alpha\right) - R_\alpha \tag{1.2}$$

$$\frac{3}{2} n_e \frac{d_e T_e}{dt} + p_e \operatorname{div} \mathbf{v}_e = -\operatorname{div} \mathbf{q}_e - \pi_e^{\alpha\beta} \frac{\partial v_e^{\alpha}}{\partial x_\beta} + Q_e$$
$$\frac{3}{2} n_i \frac{d_i T_i}{dt} + p_i \operatorname{div} \mathbf{v}_i = -\operatorname{div} \mathbf{q}_i - \pi_i^{\alpha\beta} \frac{\partial v_i^{\alpha}}{\partial x_\beta} + Q_i \tag{1.3}$$

Here n is the number of particles per unit volume, $\mathbf{v}$ the velocity, m the particle mass, p and π the pressure and viscous stress tensor, e the proton charge, $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields, T the temperature, $\mathbf{R}$ the force of interaction on the electrons by the ions, $\mathbf{q}$ the heat flux with known components. Subscript e refers to electron quantities, subscript i to ion quantities.

$$Q_e = -\mathbf{R}\mathbf{u} - \gamma(T_e - T_i), \quad Q_i = \gamma(T_e - T_i), \tag{1.4}$$
$$\gamma = 3m_e n_e / m_i \tau_e, \quad \mathbf{u} = \mathbf{v}_e - \mathbf{v}_i$$

$$\mathbf{R} = \mathbf{R}_u + \mathbf{R}_T, \quad \begin{aligned} \mathbf{R}_u &= -\alpha_\parallel \mathbf{u}_\parallel - \alpha_\perp \mathbf{u}_\perp - \alpha_\wedge \mathbf{u} \times \mathbf{h}, \qquad \mathbf{h} = \mathbf{H}/H \\ \mathbf{R}_T &= -\beta_\parallel^{uT} \nabla_\parallel T_e - \beta_\perp^{uT} \nabla_\perp T_e - \beta_\wedge^{uT} \mathbf{h} \times \nabla T_e \end{aligned} \tag{1.5}$$

$$\mathbf{q}_e^u = \beta_\parallel^{Tu} \mathbf{u}_\parallel + \beta_\perp^{Tu} \mathbf{u}_\perp + \beta_\wedge^{Tu} \mathbf{h} \times \mathbf{u}$$
$$\mathbf{q}_e = \mathbf{q}_e^u + \mathbf{q}_e^T, \quad \mathbf{q}_e^T = -\varkappa_\parallel^e \nabla_\parallel T_e - \varkappa_\perp^e \nabla_\perp T_e - \varkappa_\wedge^e \mathbf{h} \times \nabla T_e \tag{1.6}$$
$$\mathbf{q}_i = -\varkappa_\parallel^i \nabla_\parallel T_i - \varkappa_\perp^i \nabla_\perp T_i + \varkappa_\wedge^i \mathbf{h} \times \nabla T_i$$

The form of the tensor $\pi^{\alpha\beta}$ and the coefficients α, β, $\varkappa$, η are given in [1] (Formulas (4.30) to (4.45)).

The symbols $\parallel$ and $\perp$ on vectors denote the components of the vectors taken along the perpendicular to the magnetic field direction.

In the derivation of these equations, we have used the fact

$$\varepsilon_j = {}^3/_2\, n_j T_j, \qquad c_v^j = {}^3/_2$$

where ε is the internal energy per unit volume, and c_v is the specific heat per molecule.

To close the system, we must add the equations of state for the electrons and ions $p_e = n_e T_e$ and $p_i = n_i T_i$ and Maxwell's equations

$$\operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$
$$\operatorname{div} \mathbf{H} = 0, \quad \operatorname{div} \mathbf{E} = 4\pi\rho_3 \tag{1.7}$$

In what follows, without losing generality, we shall assume plasmas as being quasi-neutral, $n_e \approx n_i \approx n$.

2. Anisotropy of the transport coefficients. First of all, it is possible to simplify the expressions for the transport coefficients appearing in Formulas (4.30) to (4.45) [1], and thus, also the expressions for the fluxes **R** and **q** . From these formulas, it is clear that the coefficients η, α, β, $\varkappa$ depend on $\omega_e\tau_e$ and $\omega_i\tau_i$.

The conductivity of the medium, σ , defined by the usual formulas

$$\sigma_\parallel = e^2n^2 / \alpha_\parallel, \quad \sigma_\parallel = e^2n^2 / \alpha_\perp, \quad \sigma_\wedge = e^2n^2 / \alpha_\wedge$$

also depend on $\omega_e\tau_e$.

The dependence of the transport coefficients on $\omega_e\tau_e$ and $\omega_i\tau_i$ is called the anisotropy of the transport coefficients. As is readily seen, when $\omega_e\tau_e \ll 1$ and $\omega_i\tau_i \ll 1$, the anisotropy is not significant, and the transport coefficients are obtained from Formulas (4.30) to (4.45) of [1], with $\omega_e\tau_e = 0$ for $\omega_e\tau_e \ll 1$ and $\omega_i\tau_i = 0$ for $\omega_i\tau_i \ll 1$. When $\omega_e\tau_e \ll 1$ and $\omega_i\tau_i \ll 1$, the viscous stress tensors for the ions and electrons become particularly simple. Instead of the five viscous coefficients for the electrons and five for the ions, there remain only four coefficients in all, two for the electrons and two for the ions.

We shall clarify for which values of the macroscopic parameters the quantities $\omega_e\tau_e$ and $\omega_i\tau_i$ become small. The expressions for $\omega_e\tau_e$ and $\omega_i\tau_i$ are well known [1].

$$\tau_e = \frac{3\sqrt{m_e}\,T_e^{3/2}}{4\sqrt{2\pi}\,\lambda e^4 n_e} = \frac{3.5\cdot 10^4}{0.1\lambda}\frac{T_e^{3/2}}{n}, \qquad \omega_e = \frac{eH}{m_e c} = 1.76\cdot 10^7 H \tag{2.1}$$

$$\tau_i = \frac{3\sqrt{m_i}\,T_i^{3/2}}{4\sqrt{\pi}\,\lambda e^4 n_i} = \frac{3\cdot 10^6}{0.1\lambda}\left(\frac{m_i}{2m_p}\right)^{1/2}\frac{T_i^{3/2}}{n}, \quad \omega_i = 0.96\cdot 10^4 H\frac{m_p}{m_i}$$

$$\lambda = 23.4 - 1.15\log n + 3.45\log T_e$$

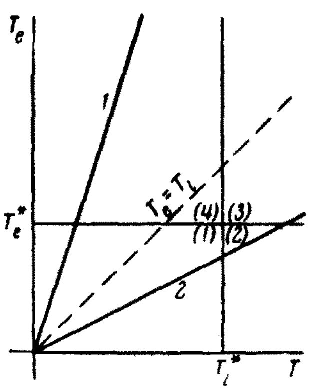

Fig. 1

The values T_e and T_i, for which $\omega_e\tau_e = 1$ and $\omega_i\tau_i = 1$, may be found from Equation (2.1) and will be denoted by T_e^* and T_i^* respectively. We can find the explicit expressions for T_e^* and T_i^* , if we neglect the influence of the dependence of λ on T_e ; this approximation is valid for a small range of variation of T_e round $T_e \approx 1$ ev.

In the T_e, T_i plane (fig.1), we draw the straight lines $T_e = T_e^*$ and $T_i = T_i^*$. This gives us four regions. In region (1), $\omega_e\tau_e < 1$ and $\omega_i\tau_i < 1$, and the anisotropy in the transport coefficients may be completely neglected. In region (2), $\omega_e\tau_e < 1$ and $\omega_i\tau_i > 1$, we may neglect the anisotropy in the transport coefficients for the electrons, while in region (4), $\omega_e\tau_e > 1$ and $\omega_i\tau_i < 1$, those for the ions. In region (3), $\omega_e\tau_e > 1$ and $\omega_i\tau_i > 1$, we must consider the anisotropy in the transport coefficients for both the electrons and the ions.

The form of the transport coefficients to be used in each of the regions (1) to (4) are given by (4.30) to (4.45) in [1], which as indicated above, simplify drastically when $\omega_e\tau_e \ll 1$ and $\omega_i\tau_i \ll 1$, and also when $\omega_e\tau_e \gg 1$ and $\omega_i\tau_i \gg 1$.

In the latter case, we must set in Formulas (4.30) to (4.45) $\omega_e\tau_e \to \infty$ and $\omega_i\tau_i \to \infty$.

We note that in the case of a one-temperature plasma

$$\omega_e\tau_e/\omega_i\tau_i = (m_i\,/\,2m_e)^{1/2}$$

Thus for $\omega_e\tau_e$ of the order unity or even greater than unity, $\omega_i\tau_i$ remains smaller than unity. In a two-temperature plasma with large T_i, the quantity $\omega_i\tau_i$ may exceed $\omega_e\tau_e$.

3. Estimation of terms in the equations of motion. Instead of variables n_e, n_i, $\mathbf{v}_e$, $\mathbf{v}_i$, it is convenient to introduce the density ρ, mean velocity $\mathbf{v}$, and current $\mathbf{j}$, as is usually done with one-temperature plasmas.

$$\rho = m_e n_e + m_i n_i, \qquad \rho\mathbf{v} = m_e n_e \mathbf{v}_e + m_i n_i \mathbf{v}_i, \qquad \mathbf{j} = en\,(\mathbf{v}_i - \mathbf{v}_e) \tag{3.1}$$

In what follows, we shall consider that $m_i\,/\,m_e \equiv m \gg 1$. (We note that the plasma equations in [1] were also written for this case). Adding Equations (1.1), and also Equations (1.2), we obtain (*) the equations for ρ and $\mathbf{v}$

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v} = 0, \qquad \frac{\partial\rho v^\alpha}{\partial t} + \operatorname{div}\rho\mathbf{v}v^\alpha = -\frac{\partial}{\partial x_\alpha}(p_e + p_i) -$$
$$-\frac{\partial}{\partial x_\beta}(\pi_e^{\alpha\beta} + \pi_i^{\alpha\beta}) + \frac{1}{c}\,\mathbf{j}\times\mathbf{H} - \operatorname{div}\rho_e\,\mathbf{u}\,u^\alpha \tag{3.2}$$

We estimate the terms appearing in Equation (3.2). To this end, we introduce the characteristic parameters: dimension L, velocity V, problem time T, characteristic problem frequency $\Omega = V/L$, current I, and also the nondimensional difference

$$|\,\mathbf{v}_i - \mathbf{v}_e\,|\,/\,V \equiv I\,/\,enV \equiv U$$

We shall consider that the order of the inertial term and that of the viscous force, the pressure term, and the diffusion term $\operatorname{div}\rho_e\mathbf{u}\,\mathbf{u}$ all do

*) We note that the sums $p_e + p_i$ and $\pi_e + \pi_i$, in general, are not equal to the total pressure and viscous stress of the mixture p and π, since in defining p_j, π_j and T_j, the random velocity of the jth component has been taken to be $\mathbf{v}_j^x = \mathbf{v}_n - \mathbf{v}_j$, rather than $\mathbf{v}^x = \mathbf{v}_n - \mathbf{v}$ (here $\mathbf{v}_n$, $\mathbf{v}_j$, $\mathbf{v}$ are respectively the true velocity of the jth type particle, the mean velocity of the jth type particle and the mean velocity of the mixture) [2].

Consideration of this difference between the viscous and thermal pressures in terms of $\mathbf{v}_j^x$ and $\mathbf{v}^x$, will be made only for extreme accuracy. This is due to the fact that the equations as now written are correct when $|\,\mathbf{v}_e - \mathbf{v}_i\,| \ll v_e{}^T$, the electron thermal velocity. We also note that if the random velocity v^x is used in defining p_j, π_j, T_j, the term $\operatorname{div}\rho_e\,\mathbf{u}\mathbf{u}^\alpha$ does not appear in the equation of motion [1 and 2].

not exceed the order of the electromagnetic force; otherwise, the influence of the electromagnetic force on the medium will be neglected. From this, it follows that

$$|\operatorname{div}\rho_e\mathbf{u}\mathbf{u}| \lesssim \frac{1}{c}|\mathbf{j}\times\mathbf{H}|, \qquad \frac{U\Omega}{\omega_i}\lesssim m$$
$$|\operatorname{div}\rho\mathbf{v}\mathbf{v}| \lesssim \frac{1}{c}|\mathbf{j}\times\mathbf{H}|, \qquad U \gtrsim \frac{\Omega}{\omega_i}, \qquad \frac{\Omega}{\omega_i}\lesssim m^{1/2} \tag{3.3}$$

We note that in the case $|\operatorname{div}\rho\mathbf{u}\mathbf{u}| \ll |[\mathbf{jH}]|/c\,(U \gg \Omega/\omega_i)$, the inertial term may be omitted in the equation of motion. In this case, the equation of motion reduces either to the magnetohydrostatic case (if viscous forces are smaller than electromagnetic forces), or to the magnetohydrodynamic Stokes equation (if the viscous forces are comparable to the electromagnetic forces).

We should clarify when the diffusion term may be neglected in the equation of motion. It is easily seen that $\rho_e\mathbf{u}\mathbf{u}/\operatorname{div}\rho\mathbf{v}\mathbf{v} = U^2m^{-1}$. When $U \ll m^{1/2}$ the diffusion term $\operatorname{div}\rho_e\mathbf{u}\mathbf{u}$ may be neglected in the equation of motion, which in this case coincides with the equation of motion in ordinary magnetohydrodynamics. When $U \sim m^{1/2}$, the diffusion and inertial terms are of the same order, and depending on the particular problem, they either both remain in the equation, or both are omitted when compared with other terms. When $U \gg m^{1/2}$, the inertial term may be omitted in the equation of motion, which then reduces either to the magnetohydrodynamic Stokes equation, or to the equation of magnetohydrostatics. For the estimation of the term $\operatorname{div}\rho_e\mathbf{u}\mathbf{u}$, it is necessary in the last two cases to compare it either with the viscous or the electromagnetic terms.

When $v_e/v_i \ll m$, $v_i \approx v$, i.e. the mean velocity is approximately equal to the mean velocity of ions. This inequality does not violate the generality; for the sake of definiteness we shall use it below.

In the derivation of Equations (1.1) to (1.3) [1], it was assumed that $|\mathbf{u}| \equiv |\mathbf{v}_e - \mathbf{v}_i| \ll v_e^T$, where v_e^T is the termal velocity of the electrons. Using this inequality, we obtain the estimate

$$|\operatorname{div}\rho_e\mathbf{u}\mathbf{u}| \sim \rho_e(\mathbf{v}_e-\mathbf{v}_i)^2/L \ll nm_e v_e^T/L \sim nT_e/L = p_e/L \tag{3.4}$$

We estimate the terms in the equation of motion, when the order of the term $|\nabla p_e| \sim p_e/L$, i.e. the change of p_e over the characteristic length is of the order of p_e.

We note that when $|\operatorname{div}\rho\mathbf{v}\mathbf{v}| \sim |\nabla p_e|$, such a large variation in p_e is possible only with significant changes in the velocity. Thus by (3.4) the term $\operatorname{div} p_e\mathbf{u}\mathbf{u}$ may be neglected in the equation of motion. In case the sum of the remaining terms on the right-hand side has the order $\operatorname{div} p_e\mathbf{u}\mathbf{u}$, then the internal term has the same order. Then to a first approximation the inertial term may also be omitted in the equation of motion. If, in addition, the viscous term is smaller than the pressure force, then the equation of motion reduces to magnetohydrostatics; if viscous and pressure forces are of the same order, it reduces to the magnetohydrodynamic Stokes equation (or to the simple hydrodynamic Stokes equation if the electromagnetic forces are smaller than the pressure forces).

It is easily seen that

$$|\operatorname{div}\rho_e\mathbf{u}\mathbf{u}| \ll c^{-1}|\mathbf{j}\times\mathbf{H}|, \qquad U\Omega/\omega_i \ll m \quad \text{when } |\nabla p_e| \sim p_e/L \tag{3.5}$$

From (3.3) and (3.5) follows $\Omega/\omega_i \ll m^{1/2}$.

It is not difficult to show that in the case when $|\nabla p_e| \sim p_e/L$, $|\nabla p_i| \sim p_i/L$, the ratio $|\operatorname{div}\pi_e|/|\nabla p_e|$ and $|\operatorname{div}\pi_i|/|\nabla p_i|$ are equal in order to the quantities $\tau_e/T \ll 1$ and $\tau_i/T \ll 1$. In other words, the viscosity may be neglected in the equation of motion in this case. As metioned above, this is valid only for sufficiently large velocities.

We compare the order of the viscous terms π_e and π_i To this end, we write the expressions for π_e^{xx}, and π_i^{xx} for definiteness; we shall do this for the case when the magnetic field is parallel to the z-axis:

$$\pi_i^{xx} = -\eta_0^i\left(\frac{1}{3}\operatorname{div}\mathbf{v} - \frac{\partial v_z}{\partial z}\right) - \eta_1^i\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right) - \eta_3^i\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\pi_e^{xx} = -\eta_0^e\left(\frac{1}{3}\operatorname{div}\mathbf{v} - \frac{\partial v_z}{\partial z} + \frac{1}{3}\operatorname{div}\mathbf{u} - \frac{\partial u_z}{\partial z}\right) - \tag{3.6}$$

$$-\eta_1^e\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial x} + \frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) - \eta_3^e\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} + \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

The remaining terms of the viscous tensors have the same order as the term π^{xx}. We note that for any $\omega_i\tau_i$ among the five ion viscosity coefficients there always exists at least one (η_0^i), whose order (cf. Section 4 of [1]) is greater or equal to the order of the other viscosity coefficients $\eta_0^i \sim n_i T_i \tau_i$. The same may be said for the electron viscosity coefficients, which in order, are smaller or equal $\eta_0^e \sim n_e T_e \tau_e$.

In what follows, we differentiate two cases. Case (A), when $U \lesssim 1$, so $\Omega / \omega_i \lesssim 1$ (3.3). Then in the expression for π_e the order of $\partial u^l / \partial x_k$ will be smaller or the same as that of $\partial v^l / \partial x_k$.

Case (B), when $U \gg 1$, in which case Ω/ω_i may be arbitrary. In this case, the order of π_e is determined by the terms $\partial u^l / \partial x_k$.

Let us estimate the order of Ω/ω_i for a typical flow of a conducting medium in a channel. Let $V = 10^5$ cm/sec , $L = 100$ cm , $H = 10^4$ gauss, then $\Omega = V/L \sim 10^3\,\mathrm{sec}^{-1}$, $\omega_i \sim 10^8\,\mathrm{sec}^{-1}$, $\Omega/\omega_i \sim 10^{-5} < 1$. From this,it is clear that in many cases of practical interest, the inequality $\Omega/\omega_i < 1$ comes true.

Comparing π_e and π_i , according to (3.6) and using Expressions (2.1) for τ_e and τ_i , we obtain that in the equation of motion the electron and ion viscosity is of the same order

$$\pi_e \sim \pi_i, \quad \text{when } T_e \sim (2m)^{1/4}\,T_i \quad (A); \qquad T_e = (2mU^{-2})^{1/4}\,T_i \quad (B)$$

The straight line 1, described by Equations $T_e = (2m)^{1/4}\,T_i$ in case (A) and $T_e = (2mU^{-2})^{1/4}\,T_i$ in case (B), is drawn in Fig.1. Below this line

$$T_e < (2m)^{1/4}\,T_i, \qquad T_e < (2mU^{-2})^{1/4}\,T_i, \qquad \pi_e < \pi_i$$

Above this line

$$T_e > (2m)^{1/4}\,T_i, \qquad T_e > (2mU^{-2})^{1/4}\,T_i, \qquad \pi_e > \pi_i$$

Consequently, in two-temperature plasmas with sufficiently high electron temperatures, the cases may arise that in the equation of motion the electron viscosity must be considered together with the ion viscosity, and sometimes it must be considered although the ion viscosity can be neglected. In the case of one-temperature plasmas, the electron viscosity is usually neglected in the equation of motion. As is clear from the estimation procedure, this is justified only when $U \ll (2m)^{1/2}$. When U is of the order of or greater than $(2m)^{1/4}$ the electron viscosity in order of magnitude may be comparable to or exceed the ion viscosity, respectively, and must be included in the equation of motion.

As shown above, the order of the viscous terms must not exceed that of the electromagnetic terms.

$$|\operatorname{div}(\pi_e + \pi_i)| \lesssim c^{-1} |\mathbf{j}\times\mathbf{H}| \tag{3.7}$$

In the case when $|\operatorname{div}(\pi_e + \pi_i)| \ll c^{-1} |\mathbf{j}\times\mathbf{H}|$, the viscous terms may be omitted in the equation of motion.

4. Estimation of terms in the generalised Ohm's law. Adding the first equation in (1.2) multiplied by $-e/m_e$ and the second equation multiplied by e/m_i, using (3.1) and taking into account $m_i / m_e \equiv m \gg 1$, we obtain an equation which is called the generalized Ohm's law

$$\frac{d\mathbf{j}}{dt} + \mathbf{j}\operatorname{div}\mathbf{v} + (\mathbf{j}\nabla)\mathbf{v} - (\mathbf{j}\nabla)\frac{\mathbf{j}}{en} = \frac{e}{m_e}\nabla p_e - \frac{e}{m_i}\nabla p_i + \frac{e}{m_e}\operatorname{div}\pi_e -$$
$$- \frac{e}{m_i}\operatorname{div}\pi_i + \frac{e^2 n}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{e}{cm_e}\mathbf{j}\times\mathbf{H} - \frac{e}{m_e}\mathbf{R}_u - \frac{e}{m_e}\mathbf{R}_T \tag{4.1}$$

The expressions for $\mathbf{R}_u$, and $\mathbf{R}_T$ are taken from (1.5).

We shall compare the order of the viscous terms

$$C_1 \equiv |\operatorname{div}\pi_e|\, e/m_e, \quad C_2 \equiv |\operatorname{div}\pi_i|\, e/m_i$$

in Ohm's law. We can show that

$$C_1 \sim C_2, \quad \text{if } (A)\ T_e \sim (2m^{-1})^{1/5}\, T_i, \qquad (B)\ T_e = (2m^{-1}\, U^{-2})^{1/5}\, T_i$$

The straight line 2, described by Equations $T_e = (2m^{-1})^{1/5} T_i$ in case (A) and $T_e = (2m^{-1}U^{-2})^{1/5} T_i$ in case (B), is shown in Fig.1. Straight lines 1 and 2 divide the quadrant into three regions, α_1, α_2 and α_3, in which

$$(\alpha_1) \qquad (2m)^{1/5}\, T_i \lesssim T_e, \qquad (2mU^{-2})^{1/5}\, T_i \lesssim T_e \tag{4.2}$$

$$(\alpha_2) \quad (2m^{-1})^{1/5}\, T_i \ll T_e \ll (2m)^{1/5} T_i, \quad (2m^{-1}U^{-2})^{1/5}\, T_i \ll T_e \ll (2mU^{-2})^{1/5}\, T_i$$

$$(\alpha_3) \qquad T_e \lesssim (2m^{-1})^{1/5}\, T_i, \qquad T_e \lesssim (2m^{-1}U^{-2})^{1/5}\, T_i$$

Thus, in Ohm's law

$$\frac{|\operatorname{div}\pi_e|}{m_e} \gg \frac{|\operatorname{div}\pi_i|}{m_i} \text{ in region } \alpha_1 + \alpha_2, \qquad \frac{|\operatorname{div}\pi_e|}{m_e} \lesssim \frac{|\operatorname{div}\pi_i|}{m_i} \text{ in region } \alpha_3 \tag{4.3}$$

In the equation of motion

$$|\operatorname{div}\pi_e| \gtrsim |\operatorname{div}\pi_i| \text{ in region } \alpha_1, \ |\operatorname{div}\pi_i| \gg |\operatorname{div}\pi_e| \text{ in region } \alpha_2 + \alpha_3 \tag{4.4}$$

We estimate the order of the viscous terms in Ohm's law (4.1).

We shall compare the terms $\operatorname{div}\pi_e / m_e$, $\operatorname{div}\pi_i / m_i$ with the term $\mathbf{j}\times\mathbf{H} / cm_e$, using Formulas (3.7) and (4.4).

In region α_3 the term $|\operatorname{div}\pi_e| \ll |\operatorname{div}\pi_i| \lesssim |\mathbf{j}\times\mathbf{H}| / c$, so

$$|\operatorname{div}\pi_i| / m_i \ll |\mathbf{j}\times\mathbf{H}|/cm_e.$$

Consequently, in Ohm's law,

$$|\operatorname{div}\pi_e| / m_e \lesssim |\operatorname{div}\pi_i| / m_i \ll |\mathbf{j}\times\mathbf{H}| / cm_e$$

and the viscous terms need not be considered.

In region α_2 the term $|\operatorname{div}\pi_e| \ll |\operatorname{div}\pi_i| \lesssim |\mathbf{j}\times\mathbf{H}|/c$, so $|\operatorname{div}\pi_e|/m_e \ll |[\mathbf{jH}]|/cm_e$. Consequently, in Ohm's law, we have

$$|\operatorname{div}\pi_i|/m_i \ll |\operatorname{div}\pi_e|/m_e \ll |\mathbf{j}\times\mathbf{H}|/cm_e$$

and the viscous terms need not be included.

In region α_1 the term $|\operatorname{div}\pi_i| \lesssim |\operatorname{div}\pi_e| \lesssim |\mathbf{j}\times\mathbf{H}|/c$, so $|\operatorname{div}\pi_i|/m_e \lesssim |\operatorname{div}\pi_e|/m_e \lesssim |\mathbf{j}\times\mathbf{H}|/cm_e$. Consequaently in Ohm's law

$$|\operatorname{div}\pi_i|/m_i \ll |\operatorname{div}\pi_e|/m_e \lesssim |\mathbf{j}\times\mathbf{H}|/cm_e$$

In other words, in regions α_2 and α_3, viscosity should not be considered in Ohm's law (4.1) In region α_1, ion viscosity should not be considered in Ohm's law (4.1). Electron viscosity in Ohm's law must be included or may be neglected in region α_1 together with the term $\mathbf{j}\times\mathbf{H}/cm_e$, if in the equation of motion viscosity is considered $(|\operatorname{div}\pi_i| \lesssim |\operatorname{div}\pi_e| \sim |\mathbf{j}\times\mathbf{H}|/c)$. Electron viscosity should not be considered in α_1 if in Equation (3.2) viscosity is not included $(|\operatorname{div}\pi_i| \lesssim |\operatorname{div}\pi_e| \ll |\mathbf{j}\times\mathbf{H}|/c)$.

The order of the inviscid terms in Ohm's law (4.1) is given as follows:

$$\begin{aligned}
&C_3 \sim \left|\frac{d\mathbf{j}}{dt}\right| \sim |\mathbf{j}\operatorname{div}\mathbf{v}| \sim (\mathbf{j}\nabla)\mathbf{v}| \sim IV/L \sim enV\Omega U\\
&C_4 \sim \left|\frac{(\mathbf{j}\nabla)\mathbf{j}}{en}\right| \sim enV\Omega U^2,\ C_5 \sim |\mathbf{R}_u|\frac{e}{m_e} \sim Ie^2n/\sigma m_e \sim neVU/\tau_e\\
&C_6 \sim \frac{e^2n}{m_e c}|\mathbf{v}\times\mathbf{H}| \sim enV\omega_e,\quad \frac{e^2n}{m_e}|\mathbf{E}| \gtrsim C_6 \qquad (4.5)\\
&C_7' \sim \frac{e}{m_e}|\nabla p_e|,\ C_7 \sim \frac{e}{cm_e}|\mathbf{j}\times\mathbf{H}| \sim enV\omega_e U,\quad C_8 \sim |\mathbf{R}_T|\frac{e}{m_e} \sim C_7'\\
&C_9' \sim \frac{e}{m_i}|\nabla p_i|,\ C_9 \sim \frac{e}{cm_i}|\mathbf{j}\times\mathbf{H}| \sim enV\omega_e U m_e/m_i,\quad C_7' \lesssim C_7
\end{aligned}$$

Using (4.5), we compare the terms in Ohm's law (4.6)

$$\frac{C_3}{C_5} \sim \tau_e/T,\qquad \frac{C_4}{C_7} \sim U\frac{\Omega m^{-1}}{\omega_i},\qquad \frac{C_5}{C_6} \sim \frac{U}{\tau_e\omega_e},\qquad \frac{C_7}{C_6} \sim U,\qquad \frac{C_9'}{C_7'} \sim \frac{T_i}{T_e}m^{-1}$$

Using (3.3), (4.6) may be written in the form

$$\frac{C_5}{C_6} \gtrsim \frac{\Omega}{\omega_i}\frac{1}{\tau_e\omega_e};\qquad \frac{C_7}{C_6} \gtrsim \frac{\Omega}{\omega_i},\qquad \frac{C_7}{C_5} \sim \omega_e\tau_e \qquad (4.7)$$

The characteristic problem time T is much greater than the electron mean collision time τ_e, thus from (4.6), it follows that $C_3 \ll C_5$. Consequently, the terms C_3 and C_2 may be omitted in Ohm's law. In the case when $|\nabla p_e| \sim p_e/L$ (inequality (3.5) comes true), the term $C_4 \ll C_7$ and may be omitted in Ohm's law; if inequalities (3.3) hold, there are cases when C_4 must be considered in Ohm's law. The term $C_9' \lesssim C_9 \ll C_7$, thus the term C_9' may be omitted in Ohm's law. Usually $T_e \gg T_i m^{-1}$ and $C_7' \gg C_9'$ (4.6); in case $T_e \lesssim T_i m^{-1}$ the term $C_7' \lesssim C_9'$ and may also be omitted in all the forms of Ohm's law given below.

5. The possible forms of Ohm's law. Using the estimates obtained above, we now give the various possible simplified forms of Ohm's law. We first consider the case, when the order of magnitude of the current (and consequently the parameter U) is unknown, but the orders Ω/ω_i and $\omega_e\tau_e$ are known. Then in comparing terms in Ohm's law, it is impossible to use the convenient estimates (4.6), and it is necessary to use estimates (4.7), which contain considerably less information than (4.6).

1. Let $\omega_e\tau_e \ll 1$. Then $C_1 \lesssim C_7 \ll C_5$, $C_4 \ll C_5$ (4.7), the anisotropy in the transport coefficients for the electron motion is absent. The following cases are possible.

1.1. When $\Omega / \omega_i \lesssim \omega_e\tau_e$; ratio C_5/C_6 may be arbitrary (4.7). Ohm's law, in general, assumes the form

$$\mathbf{j} = \sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) \tag{5.1}$$

1.2. When $\Omega / \omega_i \gg \omega_e\tau_e$; $C_5 \gg C_6$ (4.7). Ohm's law assumes the form

$$\mathbf{j} = \sigma\mathbf{E} \tag{5.2}$$

2. Let $\omega_e\tau_e \sim 1$. Then $C_1 \lesssim C_7 \sim C_5$. The following cases are possible.

2.1. When $\Omega / \omega_i \lesssim \omega_e\tau_e$; the ratio C_5/C_6 may be arbitrary (4.7). Moreover, in region α_1 the viscous term may be of the order C_5 or C_6 . The term C_4 may be of order C_5 when $U\Omega / \omega_i \sim m$ (4.6). Ohm's law takes the form

$$-\frac{m_e}{ne^2}(\mathbf{j}\nabla)\,\mathbf{j} = \nabla p_e + \operatorname{div}\pi_e + en\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{1}{c}\mathbf{j}\times\mathbf{H} - \mathbf{R}_u - \mathbf{R}_T \tag{5.3}$$

2.2 When $\Omega / \omega_i \gg \omega_e\tau_e \sim 1$; $C_5 \gg C_6$ (4.7). In region α_1 we may have $C_1 \sim C_5$. The term C_4 may be of order C_5 when $U\Omega/\omega_i \sim m$ (4.6). Ohm's law has the form

$$-\frac{m_e}{e^2n}(\mathbf{j}\nabla)\,\mathbf{j} = \nabla p_e + \operatorname{div}\pi_e + en\,\mathbf{E} - \frac{1}{c}\mathbf{j}\times\mathbf{H} - \mathbf{R}_u - \mathbf{R}_T \tag{5.4}$$

3. Let $\omega_e\tau_e \gg 1$. Then $C_7 \gg C_5$ (4.7). The terms C_6 and C_5 must be compared with C_7. The following cases are possible.

3.1. When $\Omega / \omega_i \lesssim 1$; the ratio C_7/C_6 may be arbitrary (4.7). Moreover, in region α_1 the viscous term C_1 may be of order C_7 . The term C_4 may be of order C_7 when $U\Omega/\omega_i \sim m$ according to (4.6). Ohm's law has the form

$$-\frac{m_e}{e^2n}(\mathbf{j}\nabla)\,\mathbf{j} = \nabla p_e + \operatorname{div}\pi_e + en\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) - \frac{1}{c}\mathbf{j}\times\mathbf{H} - \mathbf{R}_T \tag{5.5}$$

3.2. When $\Omega/\omega_i \gg 1$; $C_7 \gg C_6$. In the region α_1 the viscous term C_1 may be of order C_7 . The term C_4 may be of order C_7 when $U\Omega/\omega_i \sim m$ according to (4.6). Ohm's law has the form

$$-\frac{m_e}{e^2n}(\mathbf{j}\nabla)\,\mathbf{j} = \nabla p_e + \operatorname{div}\pi_e + en\mathbf{E} - \frac{1}{c}\mathbf{j}\times\mathbf{H} - \mathbf{R}_T \tag{5.6}$$

In the temperature region $\alpha_2 + \alpha_3$ the viscous term $\operatorname{div}\pi_e$ should not be considered in the Ohm's law (5.3) to (5.6). In region α_1 the electron viscosity must be considered only when it is also considered in the equation of motion. In cases when the inequality $U\Omega / \omega_i \ll m$ (3.5) holds, the term $m_e(\mathbf{j}\nabla)\,\mathbf{j}/e^2n$ should not be included in (5.3) to (5.6). In fact, for these conditions, the Ohm's law in the form of (5.3) was used in [3] to study heat exchange in fully ionized two-temperature plasma, moving in a channel with a magnetic field.

In writing Ohm's law, we have used the fact that $\mathbf{E}$ may be larger and even much larger than the term $\mathbf{v} \times \mathbf{H}/c$. If $|\mathbf{E}| \sim |\mathbf{v} \times \mathbf{H}|/c$, then those forms of Ohm's law in which the term $\mathbf{v} \times \mathbf{H}/c$ is absent, the term $\mathbf{E}$ will also be absent.

If the order of magnitude of U and $\omega_e \tau_e$ is known, then using the estimates (4.6), we may write the forms of Ohm's law in a more definite way.

4. Let $\omega_e \tau_e \ll 1$; then $C_1 \lesssim C_7 \ll C_5$, $C_4 \ll C_5$. The following cases are possible.

4.1. When $U \sim \omega_e \tau_e$; then $C_5 \sim C_6$. Ohm's law has the form (5.1).

4.2. When $U \ll w_e \tau_e$; then $C_5 \ll C_6$. Ohm's law has the form

$$\mathbf{E} = -\frac{1}{c} [\mathbf{vH}] \tag{5.7}$$

4.3. When $U \gg \omega_e \tau_e$; then $C_5 \gg C_6$. Ohm's law has the form (5.2).

5. Let $\omega_e \tau_e \sim 1$; then $C_5 \sim C_7$. The following cases are possible.

5.1. When $U \ll 1$; then $\Omega / \omega_i \ll 1$ (3.3), $C_4 \ll C_7 \ll C_6$. Ohm's law takes the form (5.7).

5.2. When $U \sim 1$; then $C_1 \lesssim C_7 \sim C_5 \sim C_6$, $C_4 \ll C_7$ (4.6). Ohm's law assumes the form (here written for $C_1 \sim C_7$)

$$0 = \nabla p_e + \operatorname{div} \pi_e + en \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H}\right) - \frac{1}{c} \mathbf{j} \times \mathbf{H} - \mathbf{R}_u - \mathbf{R}_T \tag{5.8}$$

5.3. When $U \gg 1$; then $C_1 \lesssim C_. \sim C_5 \gg C_6$. When $U\Omega / \omega_i \sim m$ (4.6), the term C_4 may be of order C_6. Ohm's law assumes the form (5.4) (here written for $C_1 \sim C_7$, $C_4 \sim C_7$).

In the cases when the viscosity is insignificant and $U\Omega / \omega_i \ll m$, the terms $\operatorname{div} \pi_e$, $m_e (\mathbf{j}\nabla)\,\mathbf{j} / e^2 n$ in Ohm's law in the last two cases must be omitted.

6. Let $\omega_e \tau_e \gg 1$; then $C_7 \gg C_5$. The following cases are possible.

6.1. When $U \ll 1$; Ohm's law reduces to the form (5.7).

6.2. When $U \sim 1$, $C_4 \ll C_7 \sim C_6$, $C_1 \lesssim C_7$. Ohm's law assumes the form (here written for $C_1 \sim C_7$)

$$0 = \nabla p_e + \operatorname{div} \pi_e + en \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H}\right) - \frac{1}{c} \mathbf{j} \times \mathbf{H} - \mathbf{R}_T \tag{5.9}$$

6.3. When $U \gg 1$; then $C_7 \gg C_6$, $C_1 \lesssim C_7$. When $U\Omega / \omega_i \sim m$ the term C_4 may be of order C_7. Ohm's law assumes the form (5.6) (in this case written for $C_1 \sim C_7$, $C_4 \sim C_7$).

In the cases when the viscosity is insignificant and $U\Omega / \omega_i \ll m$, the terms $\operatorname{div} \pi_e$ and $m_e (\mathbf{j}\nabla)\,\mathbf{j} / e^2 n$ in the Ohm's law (5.9) and (5.6) will be absent in the last two cases.

We note that in estimating the terms, it has been assumed that $C_8 \sim C_7$ ($n\nabla T_e \sim \nabla p_e$). However, the cases $C_8 \ll C_7$ and $C_8 \gg C_7$ are also possible (e.g. in a boundary layer). In the last case, the term C_8 must be compared with C_6 or with the term $e^2 nE/m_e$; when the orders are equal, C_8 must be kept (or rejected) in all the Ohm's law forms (Equations (5.1) to (5.9)) whenever these other terms are kept (or rejected).

Let $V \sim 10^5$ cm/sec , H $\sim 10^4$ gauss, variation of T_e in distance L cm of the order of 10^4 °K, $E \sim VH/c \sim 10^{-1}$ gauss$^{1/2}$ cm$^{-1/2}$ sec^{-1}. Then from $C_8 \sim C_6$ follows $E \sim 10^{-3} L^{-1}$ gauss$^{1/2}$ cm$^{-1/2}$ sec^{-1}. From this, it is obvious that for $L \sim 10^{-2}$ cm , the terms C_8 and C_6 are of same order.

If $\mathbf{E} \sim |\mathbf{v} \times \mathbf{H}|/c$, then $\mathbf{E}$ may be omitted from the Ohm's law whenever the term $\mathbf{v} \times \mathbf{H} / c$ is discarded.

From the estimation procedure, it follows that for sufficiently high electronic temperatures, in certain cases we must include in the Ohm's law terms connected with the electron viscosity, so that Ohm's law no longer remains an algebraic relation, but becomes a nonlinear differential equation.

In the case of a single-temperature plasma, the estimation of terms in Ohm's law has been carried out in [4] under the assumptions of $U \sim \Omega/\omega_i$ (inertial term and electromagnetic terms of same order) and $\pi_e \ll \pi_i$. For temperatures $T_e \sim T_i$, the present results give possible simplified forms of the equation of motion and of Ohm's law for one-temperature plasma without these additional assumptions, and thus do not agree with the results of [4].

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